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## Charged Polytropic Stars and a Generalization of Lane-Emden Equation

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In this paper we will discuss charged stars with polytropic equation of state, where we will try to derive an equation analogous to the Lane-Emden equation. We will assume that these stars are spherically symmetric, and the electric field have only the radial component. First we will review the field equations for such stars and then we will proceed with the analog of the Lane-Emden equation for a polytropic Newtonian fluid and their relativistic equivalent (Tooper, 1964)<sup>1</sup>. These kind of equations are very interesting because they transform all the structure equations of the stars in a group of differential equations which are much more simple to solve than the source equations. These equations can be solved numerically for some boundary conditions and for some initial parameters. For this we will assume that the pressure caused by the electric field obeys a polytropic equation of state too.

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It is known that the structure equations of a Newtonian star (the hydrostatic equilibrium and the mass continuity equation) can be simplified to a second order differential equation of dimensionless parameters depending on the polytropic index. This equation is much more simple and elegant than the structure equations. Once we get to a solution of the differential equation, then changing just the variables, we can get other solutions and structure data of our particular interest. This procedure was first introduced by Lane-Emden and then further developed by Chandrasekhar. In 1968, Tooper<sup>1</sup> deduced a relativistic analogous to the Lane-Emden equation with all corrections imposed by General Relativistic. Polytropic charged stars have been recently discussed in Ray et al.<sup>2</sup> where the charge distribution was assumed to vary with the mass distribution. It was showed<sup>2</sup> that in these stars the electric field must be huge ( $10^{21}$  V/m), causing an extremal instability in the star, and help the star collapse further to a charged black hole. It will be interesting to get an analogous of the Lane-Emden equation from such stars. We will try to follow the Lane-Emden and Tooper steps to get to this equation.

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## 1. Field Equations

The metric used here is the usual Schwarzschild metric,

$$ds^2 = e^{\nu(r)} c^2 dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (1)$$

The time-independent gravitational field equations reduces to:

$$e^{-\lambda} \left( -\frac{1}{r^2} + \frac{1}{r} \frac{d\lambda}{dr} \right) + \frac{1}{r^2} = -\frac{8\pi G}{c^4} \left( \rho c^2 + \frac{E^2}{8\pi} \right) \quad (2)$$

$$e^{-\lambda} \left( \frac{1}{r} \frac{d\nu}{dr} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{8\pi G}{c^4} \left( p - \frac{E^2}{8\pi} \right) \quad (3)$$

The mixed energy-momentum tensor will include the terms of the Electromagnetic Field, giving us an equation of the form

$$T_{\nu}^{\mu} = \text{diag} \left( -\rho c^2 - \frac{E^2}{8\pi}, p - \frac{E^2}{8\pi}, p + \frac{E^2}{8\pi}, p + \frac{E^2}{8\pi} \right), \quad (4)$$

where  $p$  is the pressure,  $\rho$  the mass density and  $E$  the radial electric field. The four-divergence of  $T_{\nu}^{\mu}$  should vanish for it being a conserved quantity and so we get

$$\frac{dp}{dr} = -\frac{1}{2} \frac{d\nu}{dr} (p + \rho c^2) + \frac{E}{8\pi} \left( \frac{dE}{dr} + \frac{2E}{r} \right). \quad (5)$$

This is the Tolman-Oppenheimer-Volkoff (TOV) equation for a spherically symmetric charged fluid. This equation represents the hydrostatic equilibrium for the fluid. The first term of the right hand side (r.h.s.) of equation (5) is derived from the normal matter-energy, but the second term is new. It represents the contribution from the coulombian force and the energy from the electric field. If we look at the energy-momentum tensor (equation (4)), we see that the effective pressure and density in the fluid is given for

$$p_{ef} = p - \frac{E^2}{8\pi}, \quad \rho_{ef} = \rho + \frac{E^2}{8\pi} \quad (6)$$

Now the energy-momentum tensor takes the following form:

$$T_{\nu}^{\mu} = \text{diag} \left( -\rho_{ef} c^2, p_{ef}, p + \frac{E^2}{8\pi}, p + \frac{E^2}{8\pi} \right) \quad (7)$$

We could have written the last two terms as function of the effective pressure and density, but we didn't because it is not advantageous. Now we have the modified TOV equation:

$$\frac{dp_{ef}}{dr} = -\frac{1}{2} \frac{d\nu}{dr} (p_{ef} + \rho_{ef} c^2) + \frac{E^2}{2\pi r} \quad (8)$$

## 2. The Generalization of Lane-Emden Equation

We have from the exterior Schwarzschild solution that

$$e^{-\lambda} = 1 - \frac{2GM}{rc^2} + \frac{GQ^2}{r^2 c^4}. \quad (9)$$

Usually we would have only the first term (of the mass) but here the total mass that an observer really see must include the contribution of the energy from the electric field. This one is not limited to the surface of the stars and is observed by a distant observer, so we have to include the contribution of the electric energy seen from infinity. So we can write

$$M = \int_0^{+\infty} 4\pi r^2 \left( \rho + \frac{E^2}{8\pi c^2} \right) dr, \quad Q = \int_0^R 4\pi r^2 \rho_{ch} e^{\lambda/2} dr, \quad (10)$$

where  $\rho_{ch}$  is the charge density. Now we can define an effective mass given by

$$M_{ef} = M - \frac{Q^2}{2\pi r c^2}. \quad (11)$$

With the help of the equations (9) and (11) we define  $x$  as

$$x = \frac{(1 - e^{-\lambda}) r c^2}{2GM_{ef}}. \quad (12)$$

With this we rewrite  $e^{-\lambda}$  in the following form:

$$e^{-\lambda} = 1 - \frac{2GM_{ef}x}{rc^2}. \quad (13)$$

Writing equation (2) as function of  $x$  and using equation (13) we get the following relation:

$$M_{ef} \frac{dx}{dr} = 4\pi r^2 \rho_{ef}. \quad (14)$$

Now we have to write the explicit relation between effective pressure and effective density, then use in the modified TOV equation (8). We will assume that this relation is given by the following polytropic equation:

$$p_{ef} = K_{efe} \rho_{ef}^{1+1/n}. \quad (15)$$

If we do a dimensional analysis we will see that all terms of the r.h.s. have the dimension of pressure divided by unit of length, which is not surprising since the left hand side is a derivative of the pressure with respect to the radius. In this sense we will call the second term of the r.h.s. in the following form :

$$\frac{E^2}{2\pi r} = \frac{dp_{el}}{dr} \quad (16)$$

where  $p_{el}$  is a pressure related to the electric field. And in order to make further progress we will suppose that this pressure also obeys a polytropic equation of state

$$p_{el} = K_{el} \rho_{el}^{1+1/n}. \quad (17)$$

Now let us write the density in a parametric form

$$\rho_{ef} = \rho_{efc} \theta^n, \quad \rho_{el} = \rho_{elc} \theta^n, \quad (18)$$

where  $\rho_{efc}$  is the central effective pressure and  $\rho_{elc}$  is the central electric pressure. In this notation, both electric and effective pressure take the form

$$p_{ef} = K_{efe} \rho_{efc}^{1+1/n} \theta^{n+1}, \quad p_{el} = K_{el} \rho_{elc}^{1+1/n} \theta^{n+1}. \quad (19)$$

Using these modifications on equation (8) we get

$$\frac{d\nu}{dr} = -2(n+1) \frac{(\sigma - \sigma_{el}\eta)}{\sigma\theta + 1} \frac{d\theta}{dr}, \quad (20)$$

where

$$\sigma = \frac{K\rho_{efc}^{1/n}}{c^2}, \quad \sigma_{el} = \frac{K\rho_{elc}^{1/n}}{c^2}, \quad \eta = \frac{\rho_{elc}}{\rho_{efc}}. \quad (21)$$

Integrating equation (20) and using the fact that  $\nu \rightarrow \nu_c$  when  $\theta \rightarrow 1$  we get

$$e^\nu = e^{\nu_c} \left( \frac{\sigma + 1}{\sigma\theta + 1} \right)^{2\left(\frac{n+1}{\sigma}\right)(\sigma - \sigma_{el}\eta)}. \quad (22)$$

Using equation (22), when  $r = R$  we have  $\rho \rightarrow 0$  and  $\theta = 0$ , and comparing equation (9) we get

$$e^\nu = (\sigma\theta + 1)^{-2\frac{n+1}{\sigma}(\sigma - \sigma_{el}\eta)} \left( 1 - \frac{2GM_{ef}}{rc^2} \right). \quad (23)$$

Using equation (20) together with the field equation (3), we get a new differential equation connecting  $x$  and  $\theta$

$$\frac{(n+1)(\sigma - \sigma_{el}\eta)}{\sigma\theta + 1} r \frac{d\theta}{dr} \left( 1 - \frac{2GM_{ef}x}{rc^2} \right) + \frac{GM_{ef}x}{rc^2} + \frac{G\sigma M_{ef}}{c^2} \frac{dx}{dr} \theta = 0. \quad (24)$$

Using this together with equation (14) and the following changing of variables

$$r = \frac{\xi}{A}, \quad v(\xi) = \frac{A^3 M_{ef} x}{4\pi\rho_{ef}}, \quad A = \left[ \frac{4\pi G\rho_{efc}}{(n+1)K\rho_{efc}^{1/n}} \right]^{1/2}, \quad (25)$$

we get the following differential equations:

$$\xi^2 \frac{d\theta}{d\xi} \frac{(1 - \frac{\sigma_{el}\eta}{\sigma}) - (n+1)(\sigma - \sigma_{el}\eta)v/\xi}{\sigma\theta + 1} + v + \sigma\theta\xi \frac{dv}{d\xi} = 0; \quad (26)$$

$$\frac{dv}{d\xi} = \xi^2 \theta^n. \quad (27)$$

These are the General Lane-Emden equation for a spherically symmetric charged polytropic fluid. Solving these equation we can step back and find those variables of our interest. Numerical solution and verification of these equations with the results obtained by solving the TOV in the *normal* procedure is beyond the scope of this present paper and will be shown in a future article.

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